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Saddle Points and Pareto Points in Multiple Objective Programming

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Abstract. In this paper relationships between Pareto points and saddle points are studied in convex and nonconvex multiple objective programming. The analysis is based on partitioning the index sets of objectives and constraints and splitting the original problem into subproblems having a special structure. The results are based on scalarizations of multiple objective programs and related linear and augmented Lagrangian functions. In the nonconvex case, a saddle point characterization of Pareto points is possible under assumptions that guarantee existence of Pareto points and stability conditions of single objective problems. Essentially, these conditions are not stronger than those in analogous results for single objective programming.

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1. Introduction

Many researchers have contributed to the theory and methodology of multiple objective programming. In particular, a lot of attention has been given to the development of various conditions for Pareto solutions. Among a great deal of studies, one specific direction has been to relate Pareto solutions to saddle points resulting from a duality framework associated with the multiple objective program (MOP) of interest. Due to their distinctive features, saddle points have been of special interest in single objective nonlinear programming. Over the years, obtained results have been carried over and extended for MOPS. The available literature on various types of duality in multiple objective programming is very rich and there are numerous papers dealing specifically with saddle points in various

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settings (e.g., infinite dimensional spaces, generalized convexity, approximate solutions, theory of games, nondifferentiable problems, etc.). Many duality results for vector-valued Lagrangian functions associated with general MOPS defined with convex cones are contained in Sawaragi et al. [16]. Valyi [21] extended saddle-point conditions for convex problems to problems with approximate solutions. Those results were generalized by Breckner et al. [3] for vector approximation problems. Similarly, approximate saddle-point results were extended by Rong and Wu [15] for problems with set-valued maps and by Dutta and Vetrivel [6] for problems with generalized approximate solutions. A vector-valued generalized Lagrangian was constructed and analyzed by lacob [11], Singh et al. [17], and others. A set-valued mapping was used by Huang and Yang [10] to construct and examine a generalized augmented Lagrangian.

A different direction was undertaken by those who studied scalar-valued Lagrangians associated with scalarizations of MOPs. Van Rooyen et al. [22] constructed a Lagrangian function for scalarized convex MOPs and developed a saddle-point condition for Pareto solutions, which is both necessary and sufficient. TenHuisen and Wiecek [18] proposed a framework for developing generalized Lagrangian-type scalarizing functions for nonconvex programs. They used the augmented function to develop solution approaches to finding Pareto points for bicriteria programs [19] and multiple criteria programs [20]. The purpose of this article is to further examine relationships between Pareto solutions and saddle points within the framework of scalarized multiple objective programming in finite dimensions. As we examine saddle points not only for convex but also for nonconvex problems, we apply the augmented (quadratic) Larangian function originally proposed in single objective nonlinear programming.

In this paper we consider the general MOP:

$$\begin{aligned} &\min\{\Phi^q(x), q \in \mathcal{Q}\} \\ &\text{subject to } f^j(x) \leqslant 0, \quad j \in \mathcal{P}, \end{aligned} \qquad \text{(MOP)}$$

where

$$F := \{x \in \mathbb{R}^n : f^j(x) \leqslant 0, \quad j \in \mathcal{P}\}$$

is the feasible set, the functions $\Phi^q(x), q \in, \mathcal{Q}, f^j(x), j \in \mathcal{P}$, are all real-valued, and $\mathcal{Q} := \{1, \ldots, Q\}$ denotes the index set of the objective functions and $\mathcal{P} := \{1, \ldots, m\}$ denotes the index set of the constraints. The convex hull of a set S will be denoted by conv S.

A feasible point $\hat{x} \in F$ is called a global Pareto solution (or a global efficient solution) for (MOP), if there is no other point $x \in F$ such that $\Phi^i(x) \leqslant \Phi^i(\hat{x})$ for all $i \in \mathcal{Q}$ and $\Phi^j(x) < \Phi^j(\hat{x})$ for some j. A feasible point $\hat{x} \in F$ is said to dominate a point $x \in F$ if $\Phi^i(\hat{x}) \leqslant \Phi^i(x)$ for all $i \in \mathcal{Q}$ and $\Phi^j(\hat{x}) = \Phi^j(x)$ for some j. If $\hat{x}$ is a global Pareto point, the corresponding point $\Phi(\hat{x}) = (\Phi^1(\hat{x}), \ldots, \Phi^Q(\hat{x}))$ in the objective space is called globally nondominated. For nonconvex problems, the concept of local Pareto (nondominated) points is essential. A point $x^* \in F$ is a local Pareto point, if there exists a neighbourhood $N(x^*)$ of x^* such that x^* is a Pareto point in $F \cap N(x^*)$. In this study we focus on global Pareto solutions and refer to them as Pareto solutions.

We make extensive use of the scalarization of (MOP) introduced by Charnes and Cooper [5] and formulated for x^*, an arbitrary feasible point of (MOP):

$$\begin{aligned} \min \quad & \sum_{q=1}^{Q} \Phi^q(x) \\ \text{subject to } & \Phi^q(x) - \Phi^q(x^*) \leqslant 0, \quad q = 1, \ldots, Q, \\ & x \in F. \end{aligned} \qquad (\mathrm{CC}(\mathrm{MOP}, x^*))$$

As this scalarization depends on the point x^*, we refer to this problem as (CC(MOP, x^*)). It is also well known that problem (CC(MOP, x^*)) provides a method for finding Pareto points, as Charnes and Cooper observed [5]: A point $x^* \in F$ is a Pareto solution for (MOP) if and only if x^* is an optimal solution for problem (CC(MOP, x^*)).

In Section 2, we partition the index set of the objective functions and the index set of the constraints. The idea of partitioning these sets, that appeared in Zlobec [24] and was later explored in [22] and [12], was the inspiration for this paper. We propose two partitions, one for feasible points only while in the other we allow infeasible points, i.e., we relax feasibility. These partitions determine a framework within which we study relationships between Pareto points and saddle points of convex and nonconvex MOPs. In Section 3, we derive a saddle point characterization of Pareto points for convex programs. Although in this section we follow upon the results in [22], we introduce different index sets to define the Lagrangian functions which makes our results easier to handle. In Section 4, we analyze nonconvex programs and derive a saddle point characterization of Pareto points by applying the augmented Lagrangian function. Final conclusions are contained in Section 5.

2. Partitioning the Index Set of Objective Functions

Given a feasible point x^* of (MOP) and the objective function values at this point, one may be interested whether it is still possible to improve the values of some criteria while the other criteria do not deteriorate. To answer this question, it is convenient to introduce the set of points which allow improvement of some objective functions with respect to a given point x^*. We can do this with respect to the feasible set F or the whole space $\mathbb{R}^n$ and define

$$S^{\leqslant}(x^*) := \{x \in S : \Phi^i(x) \leqslant \Phi^i(x^*), i \in \mathcal{Q}\}, \tag{1}$$

where $S \in \{F, \mathbb{R}^n\}$. To indicate which definition of S is used, we shall write $S_F^{\leqslant}(x^*)$ and $S_{\mathbb{R}^n}^{\leqslant}(x^*)$. Using the concept of level sets containing the point x^*,

$$L^i_{\leqslant}(x^*) := \{x \in F : \Phi^i(x) \leqslant \Phi^i(x^*)\},$$

we have that $S_F^{\leqslant}(x^*)$ is equal to the intersection of all the level sets:

$$S_F^{\leqslant}(x^*) = \bigcap_{i=1}^{Q} L^i_{\leqslant}(x^*).$$

In [8] it was shown that a point $x^* \in F$ is a Pareto point of (MOP) if and only if

$$\bigcap_{i=1}^{Q} L^i_{\leqslant}(x^*) = \bigcap_{i=1}^{Q} L^i_{=}(x^*), \tag{2}$$

where $L^i_{=}(x^*)$ denotes the level curve of the objective Φ^i passing through x^*:$L^i_{=}(x^*)$:=$\{x \in F : \Phi^i(x) = \Phi^i(x^*)\}$. Clearly, if x^* is not a Pareto point these intersections must be different. As it is desirable to distinguish between the objective functions that allow improvement and those that do not, we partition the index set $\mathcal{Q}$ with respect to $S^{\leqslant}(x^*)$:

$$\mathcal{Q}^{=}(x^*) := \{i \in \mathcal{Q} : x \in S^{\leqslant}(x^*) \Rightarrow \Phi^i(x) = \Phi^i(x^*)\} \tag{3}$$

and

$$\mathcal{Q}^{<}(x^*) := \{i \in \mathcal{Q} : \exists x \in S^{\leqslant}(x^*) \text{ such that } \Phi^i(x) < \Phi^i(x^*)\}. \tag{4}$$

Obviously, we have $\mathcal{Q} = \mathcal{Q}^{=}(x^*) \cup \mathcal{Q}^{<}(x^*)$. If $S = F$, only feasible points are considered in the definition of $S^{\leqslant}(x^*), \mathcal{Q}^{=}(x^*)$, and $\mathcal{Q}^{<}(x^*)$. This partition is called the *feasible point partition* of the index set of the objective functions. If, however, $S = \mathbb{R}^n$ we include infeasible points in the definition of $S^{\leqslant}(x^*)$, $\mathcal{Q}^{=}(x^*)$, and $Q^{\leqslant}(x^*)$, too. We call the resulting partition the *infeasible point partition* of the index set of the objective functions. Throughout the paper we use indices F and I to indicate to which partition we refer, i.e., $\mathcal{Q}_F^{=}(x^*)$ and $\mathcal{Q}_F^{<}(x^*)$ are defined with respect to $S_F^{\leqslant}(x^*)$ while $\mathcal{Q}_I^{=}(x^*)$ and $\mathcal{Q}_I^{<}(x^*)$ are defined with respect to $S_{\mathbb{R}^n}^{\leqslant}(x^*)$. Statements where no index is present pertain to both partitions.

Given a point $x^* \in S$ which is not a Pareto point for (MOP), we shall show that one does not have to consider all the objective functions to find a Pareto point and can restrict the optimization to the objective functions in $\mathcal{Q}^{<}(x^*)$. We therefore formulate a MOP with a smaller number of objective functions and refer to it as (MOP($\mathcal{Q}^{<}(x^*)$)), as this problem depends on the point x^*:

$$\begin{aligned} \min \quad & \{\Phi^q(x), q \in \mathcal{Q}^{<}(x^*)\} \\ \text{subject to } & x \in S \\ & \Phi^i(x) \leqslant \Phi^i(x^*), \quad i \in \mathcal{Q}. \end{aligned} \tag{MOP($\mathcal{Q}^{<}(x^*)$)}$$

The feasible set of (MOP($\mathcal{Q}^{<}(x^*)$)) is just $S^{\leqslant}(x^*)$. It turns out that the Pareto points for subproblem (MOP($\mathcal{Q}^{<}(x^*)$)) are closely related to the Pareto points for the original problem (MOP).

THEOREM 1. *Let x^* be a feasible point of* (MOP) *which is not a Pareto point. Then the following statements hold*:

1. *A point $\bar{x} \in S_F^{\leqslant}(x^*)$ is a Pareto point for* (MOP($\mathcal{Q}_F^{<}(x^*)$)) *if and only if $\bar{x}$ is a Pareto point for* (MOP).
2. *If $\hat{x} \in S_{\mathbb{R}^n}^{\leqslant}(x^*)$ is a Pareto point for* (MOP($\mathcal{Q}_I^{<}(x^*)$)) *and feasible for* (MOP) *then it is also a Pareto point for* (MOP).

Proof. We prove the first statement only. The second statement is analogous to the only if part of the first.

($\Rightarrow$) Let $\bar{x}$ be a Pareto point of (MOP($\mathcal{Q}_F^{<}(x^*)$)). Assume that $\bar{x}$ is not a Pareto point for (MOP). Then there is an $x' \in F$ such that

$$\Phi^i(x') \leqslant \Phi^i(\bar{x}) \qquad \forall i \in \mathcal{Q},$$
$$\Phi^k(x') < \Phi^k(\bar{x}) \quad \text{for some } k \in \mathcal{Q}.$$

Because $\bar{x} \in S_F^{\leqslant}(x^*)$ we have also that $\Phi^i(\bar{x}) \leqslant \Phi^i(x^*)$ for all $i \in \mathcal{Q}$ and therefore $x' \in S_F^{\leqslant}(x^*)$. From $\Phi^k(x') < \Phi^k(x^*)$ we get $k \in \mathcal{Q}_F^{<}(x^*)$. In par-tic-ular, we conclude that there exists an $x' \in S_F^{\leqslant}(x^*)$ such that

$$\Phi^i(x') \leqslant \Phi^i(\bar{x}) \qquad \forall i \in \mathcal{Q}_F^{<}(x^*),$$
$$\Phi^k(x') < \Phi^k(\bar{x}) \quad \text{for some } k \in \mathcal{Q}_F^{<}(x^*),$$

which contradicts the fact that $\bar{x}$ is a Pareto solution for (MOP($\mathcal{Q}_F^{<}(x^*)$)).

($\Leftarrow$) Let $\bar{x} \in S_F^{\leqslant}(x^*)$ be a Pareto point of (MOP). Assume that $\bar{x}$ is not a Pareto point for (MOP($\mathcal{Q}_F^{<}(x^*)$)). Then there is an $x' \in S_F^{\leqslant}(x^*)$ such that

$$\Phi^q(x') \leqslant \Phi^q(\bar{x}) \quad \forall q \in \mathcal{Q}_F^{<}(x^*),$$
$$\Phi^k(x') < \Phi^k(\bar{x}) \quad \text{for some } k \in \mathcal{Q}_F^{<}(x^*).$$

Using the definition of $\mathcal{Q}_F^{=}(x^*)$ we have that $\Phi^i(\bar{x}) = \Phi^i(x')$, for all $i \in \mathcal{Q}_F^{=}(x^*)$ so we conclude that there is an $x' \in S_F^{\leqslant}(x^*)$ such that

$$\Phi^q(x') \leqslant \Phi^q(\bar{x}) \quad \forall q \in \mathcal{Q},$$
$$\Phi^k(x') < \Phi^k(\bar{x}) \quad \text{for some } k \in \mathcal{Q},$$

which contradicts the fact that $\bar{x}$ is Pareto for (MOP). □

We note that a similar observation based on a different partition of the index set of the objective functions for convex MOPs was made in [12]. Note also that from the definition of $\mathcal{Q}_F^{<}(x^*)$, if $x^* \in F$ is a Pareto solution for (MOP) then (MOP($\mathcal{Q}_F^{<}(x^*)$)) is not defined.

Given a point $x^* \in F$ which is not a Pareto point for (MOP) and a feasible point $\bar{x} \in F$, a variation of (MOP($\mathcal{Q}^{<}(x^*)$)) can be formulated:

$$\begin{aligned}\min \quad & \{\Phi^q(x), \quad q \in \mathcal{Q}^<(x^*)\} \\ \text{subject to } & x \in S \\ & \Phi^i(x) \leqslant \Phi^i(\bar{x}), \quad i \in \mathcal{Q}^=(x^*).\end{aligned} \qquad (\text{MOP}(\mathcal{Q}^<(x^*), \bar{x}))$$

It turns out that every Pareto point for this reduced problem is also Pareto for (MOP). The converse also holds for the feasible point partition.

THEOREM 2. *Let* x^* *be a feasible point of* (MOP) *which is not a Pareto point and let* $\bar{x} \in F$. *Then*

1. *If* $\bar{x}$ *dominates* x^* *and is a Pareto point for* (MOP($\mathcal{Q}^<(x^*), \bar{x}$)), *then* $\bar{x}$ *is also a Pareto point for* (MOP).
2. *If* $S = F$ *and* $\bar{x}$ *is a Pareto point for* (MOP) *then it is also a Pareto point for* (MOP($\mathcal{Q}_F^{\leqslant}(x^*), \bar{x}$)).

Proof. 1. Assume that a point $\bar{x}$ dominating x^* is Pareto optimal for problem (MOP($\mathcal{Q}^<(x^*), \bar{x}$)), but not for the original (MOP). Then there exists an $x' \in F$ such that $\Phi^q(x') \leqslant \Phi^q(\bar{x})$ for all $q \in \mathcal{Q}$ and $\Phi^k(x') < \Phi^k(\bar{x})$ for some k.

Now we consider two cases: $k \in \mathcal{Q}^<(x^*)$ and $k \in \mathcal{Q}^=(x^*)$. In the former, we have $x' \in F$ such that

$$\begin{aligned}&\Phi^q(x') \leqslant \Phi^q(\bar{x}) \quad \forall q \in \mathcal{Q}^<(x^*),\\ &\Phi^k(x') < \Phi^k(\bar{x}) \quad \text{for some } k \in \mathcal{Q}^<(x^*),\\ &\Phi^q(x') \leqslant \Phi^q(\bar{x}) \quad \forall q \in \mathcal{Q}^=(x^*).\end{aligned}$$

Therefore x' is feasible for (MOP($\mathcal{Q}^<(x^*), \bar{x}$)) and the above inequalities contradict Pareto optimality of $\bar{x}$ for this problem.

In the latter, we have $x' \in F$ such that

$$\begin{aligned}&\Phi^q(x') \leqslant \Phi^q(\bar{x}) \quad \forall q \in \mathcal{Q}^=(x^*),\\ &\Phi^k(x') < \Phi^k(\bar{x}) \quad \text{for some } k \in \mathcal{Q}^=(x^*),\\ &\Phi^q(x') \leqslant \Phi^q(\bar{x}) \quad \forall q \in \mathcal{Q}^<(x^*).\end{aligned}$$

From the fact that $\bar{x}$ dominates x^* we also know that $\bar{x}$ (and therefore x') belong to the set $S^{\leqslant}(x^*)$. This implies that $\Phi^k(x') = \Phi^k(x^*)$, since $k \in \mathcal{Q}^=(x^*)$. Combining this equality with the strict inequality above, we get

$$\Phi^k(x^*) = \Phi^k(x') < \Phi^k(x'),$$

which contradicts the fact that $\bar{x}$ dominates x^*.

2. Let $\bar{x} \in F$ be a Pareto point for (MOP). Assume that $\bar{x}$ is not Pareto for (MOP($\mathcal{Q}_F^{\leqslant}(x^*), \bar{x}$)). Then there is an $x' \in F$ such that

$$\begin{aligned}&\Phi^i(x') \leqslant \Phi^i(\bar{x}) \quad \forall i \in \mathcal{Q}_F^{\leqslant}(x^*),\\ &\Phi^k(x') < \Phi^k(\bar{x}) \quad \text{for some } k \in \mathcal{Q}_F^{\leqslant}(x^*).\end{aligned}$$

Since x' is feasible for $(\mathrm{MOP}(\mathcal{Q}_F^{<}(x^*), \bar{x}))$ then $\Phi^q(x') \leqslant \Phi^q(\bar{x})$ for all $q \in \mathcal{Q}_F^{=}(x^*)$. The three inequalities above imply that $\bar{x}$ is not Pareto for (MOP). □

In order to find a Pareto point for subproblem $(\mathrm{MOP}(\mathcal{Q}^{<}(x^*)))$ we investigate the scalarization:

$$\begin{array}{lll} \min & \sum_{q \in \mathcal{Q}^{<}(x^*)} \Phi^q(x) & \\ \text{subject to} & \Phi^q(x) - \Phi^q(\hat{x}) \leqslant 0, \quad q \in \mathcal{Q}^{<}(x^*), & (\mathrm{CC}(\mathrm{MOP}(\mathcal{Q}^{<}(x^*)), \hat{x})) \\ & x \in S^{\leqslant}(x^*), & \end{array}$$

where $\hat{x} \in S^{\leqslant}(x^*)$ is an arbitrary feasible point of $(\mathrm{MOP}(\mathcal{Q}^{<}(x^*)))$. Again, by the Charnes and Cooper result, there is a close interrelation between Pareto points of $(\mathrm{MOP}(\mathcal{Q}^{<}(x^*)))$ and optimal solutions of its scalarization: Let x^* be a feasible point of (MOP) which is not a Pareto point and let $\hat{x} \in S^{\leqslant}(x^*)$. If $\hat{x}$ is an optimal solution of $(\mathrm{CC}(\mathrm{MOP}(\mathcal{Q}_I^{<}(x^*)), \hat{x}))$ and feasible for (MOP), then it is also a Pareto solution for (MOP). As usual, for the feasible point partition the result is stronger, and we have coincidence of the two Pareto sets.

A similar result can be obtained using Theorem 2 in combination with the Charnes and Cooper scalarization of $(\mathrm{MOP}(\mathcal{Q}_F^{<}(x^*), \bar{x}))$: Let $x^* \in F$ be a feasible point of (MOP) which is not a Pareto point and let $\bar{x} \in S_F^{\leqslant}(x^*)$ be a point dominating x^*. Then $\bar{x}$ is an optimal solution of the Charnes and Cooper scalarization of $(\mathrm{MOP}(\mathcal{Q}_F^{<}(x^*)), \bar{x})$ if and only if $\bar{x}$ is a Pareto point of (MOP).

An analogous partition, complementing the partition of objectives, can be introduced for the index set of the constraints:

$$\mathcal{P}^{=}(x^*) := \{j \in \mathcal{P} : x \in S^{\leqslant}(x^*) \Rightarrow f^j(x) = 0\}$$

and

$$\mathcal{P}^{<}(x^*) := \{j \in \mathcal{P} : \exists x \in S^{\leqslant}(x^*) \text{ such that } f^j(x) < 0\},$$

thus $\mathcal{P} = \mathcal{P}^{=}(x^*) \cup \mathcal{P}^{<}(x^*)$. Here we simply distinguish between the active and inactive constraints with respect to the set $S^{\leqslant}(x^*)$. The partition with respect to $S_F^{\leqslant}(x^*)$ is called the *feasible point partition* of the index set of the constraints and denoted by $\mathcal{P}_F^{=}(x^*)$ and $\mathcal{P}_F^{<}(x^*)$. The partition with respect to $S_{\mathbb{R}^n}^{\leqslant}(x^*)$ is called the *infeasible point partition* of the index set of the constraints and denoted by $\mathcal{P}_I^{=}(x^*)$ and $\mathcal{P}_I^{<}(x^*)$. In particular, when $S = F$ and $S_F^{\leqslant}(x^*) = \{x^*\}$, i.e., when x^* is Pareto for (MOP), the feasible point partition is the partition into active and inactive constraints at x^*.

To conclude this section we remark that the sets $\mathcal{Q}^{<}(x^*)$ and $\mathcal{P}^{<}(x^*)$ are directly based on the concept of level sets of all the objective functions considered simultaneously (the former can be easily determined by just comparing the functions' values at x^* with these functions' constrained minima). The index sets used in [22] are defined by means of level sets of

$(Q-1)$-element subsets of the set of all objective functions and therefore require more complicated calculations.

3. Convex Problems

In this section we investigate the special case of convex MOPs, i.e. we assume that all the functions $\Phi^i, i \in \mathcal{Q}$ and $f^j, j \in \mathcal{P}$ are convex in (MOP). In particular, the feasible set F is convex. Note that if (MOP) is a convex problem then $(\mathrm{MOP}(\mathcal{Q}^<(x^*)))$ is also convex: For the feasible point partition, $S_F^{\leqslant}(x^*)$ is the intersection of the (convex) level sets $L^i_{\leqslant}(x^*)$, for the infeasible point partition, it is the intersection of the convex sets $\{x \in \mathbb{R}^n : \Phi^i(x^*) \leqslant \Phi^i(x^*), i \in \mathcal{Q}\}$.

In this section we use both the feasible and infeasible point partitions and prove saddle point results for Pareto points of the convex (MOP). Some of these results have first been obtained by the first author in his Ph.D. thesis [7]. We follow upon the study of convex MOPs performed in [24], [22], and [12], and obtain similar results using different partitions of the index sets.

Using the partition (3), (4) of the index set of the objective functions defined in Section 2, we observe a condition for a point $x^* \in F$ not to be Pareto. An analogous observation has been made in [22].

A point $x^* \in F$ is not a Pareto point for (MOP) if both $Q^<(x^*) \neq \emptyset$ and there exists a point $\bar{x} \in F$ such that

$$\Phi^i(\bar{x}) < \Phi^i(x^*) \quad \forall i \in \mathcal{Q}^<(x^*),$$
$$\Phi^i(\bar{x}) = \Phi^i(x^*) \quad \forall i \in \mathcal{Q}^=(x^*).$$

More significantly, for the feasible point partition equivalence holds.

If we think of $(\mathrm{MOP}(\mathcal{Q}_F^<(x^*)))$ in the context of convex problems, we can strengthen Theorem 1. If x^* is not a Pareto point for (MOP), then by the above observation, there exists a Pareto point $\bar{x}$ for $(\mathrm{MOP}(\mathcal{Q}_F^<(x^*)))$ such that $\Phi^i(\bar{x}) = \Phi^i(x^*)$ for all $i \in \mathcal{Q}_F^=(x^*)$ and $\Phi^i(\bar{x}) < \Phi^i(x^*)$ for all $i \in \mathcal{Q}_F^<(x^*)$, and furthermore, due to Theorem 1, $\bar{x}$ is a Pareto point for (MOP).

When we use the feasible point partition, the geometrical characterization of Pareto solutions given by (2) implies that a point $x^* \in F$ is Pareto if and only if $\mathcal{Q}_F^<(x^*)$ is empty. This conclusion is valid for general (nonconvex) problems. For a non-Pareto point x^* the above observation allows a distinction between convex and nonconvex problems. The existence of the point $\bar{x}$ which satisfies all the conditions of the observation simultaneously cannot be shown in the general (nonconvex) case, i.e. in the general context of [8]. The implication for the convex (MOP) is that x^* not being a Pareto point implies the existence of a feasible point $\bar{x}$ for which $\Phi^i(\bar{x}) < \Phi^i(x^*)$ for all $i \in \mathcal{Q}^<(x^*)$ *simultaneously*.

Including the partition of the index set of the constraints, $\mathcal{P}^{<}(x^*)$ and $\mathcal{P}^{=}(x^*)$, we extend the above observation and obtain a more specific result concerning the constraints, analogous to a result in [22].

A point $x^* \in F$ is not a Pareto point for (MOP) if both $\mathcal{Q}^{<}(x^*) \neq \emptyset$ and there exists a point $\bar{x} \in F$ such that

$$\Phi^i(\bar{x}) < \Phi^i(x^*) \quad \forall i \in \mathcal{Q}^{<}(x^*), \tag{5}$$

$$\Phi^i(\bar{x}) = \Phi^i(x^*) \quad \forall i \in \mathcal{Q}^{=}(x^*), \tag{6}$$

$$f^j(\bar{x}) < 0 \quad \forall j \in \mathcal{P}^{<}(x^*), \tag{7}$$

$$f^j(\bar{x}) = 0 \quad \forall j \in \mathcal{P}^{=}(x^*). \tag{8}$$

Again, for the feasible point partition, equivalence holds.

For a feasible point $x^* \in F$ define now the set

$$\begin{aligned} \bar{S}^{\leqslant}(x^*) := &\{x \in \mathbb{R}^n : \Phi^i(x) \leqslant \Phi^i(x^*) \quad \forall i \in \mathcal{Q}^{=}(x^*)\} \cap \\ &\{x \in \mathbb{R}^n : f^j(x) \leqslant 0 \quad \forall j \in \mathcal{P}^{=}(x^*)\}, \end{aligned} \tag{9}$$

i.e., $\bar{S}^{\leqslant}(x^*)$ is the set of points which satisfy the constraints in $\mathcal{P}^{=}(x^*)$ and allow a decrease for objectives in $\mathcal{Q}^{=}(x^*)$. We shall again use subscripts F and I to indicate to which partition we refer, if necessary. Note that these points are not necessarily contained in F. Therefore, $\overline{S}^{\leqslant}(x^*)$ is not necessarily contained in $S_F^{\leqslant}(x^*)$ or vice versa. Since only $\mathcal{Q}^{=}$ and $\mathcal{P}^{=}$ are used in definition (9) the same applies to the comparison of $\overline{S}_{\mathbb{R}^n}^{\leqslant}(x^*)$ and $S_{\mathbb{R}^n}^{\leqslant}(x^*)$.

Given a feasible point $x^* \in F$, define a *Lagrangian function*

$$L(x, \lambda, \mu) := \sum_{i \in \mathcal{Q}^{<}(x^*)} \lambda_i \Phi^i(x) + \sum_{j \in \mathcal{P}^{<}(x^*)} \mu_j f^j(x) \tag{10}$$

associated with the problem

$$\begin{aligned} &\min \quad \{\Phi^q(x), q \in \mathcal{Q}^{<}(x^*)\} \\ &\text{subject to } f^j(x) \leqslant 0, \quad j \in \mathcal{P}^{<}(x^*) \\ &\qquad x \in \bar{S}^{\leqslant}(x^*), \end{aligned} \tag{11}$$

which, using definition (9), is equivalent to

$$\begin{aligned} &\min \quad \{\Phi^q(x), q \in \mathcal{Q}^{<}(x^*)\} \\ &\text{subject to } \Phi^q(x) \leqslant \Phi^q(x^*), \quad q \in \mathcal{Q}^{=}(x^*) \\ &\qquad x \in F. \end{aligned} \tag{12}$$

Observe that the structure of (MOP($\mathcal{Q}^{<}(x^*), \bar{x}$)) is similar to that of (12), however the important difference is that in the former the point x^* is assumed not to be Pareto while in the latter it is meant to be Pareto. Note also that the constraint $x \in F$ is present no matter if $S = F$ or $S = \mathbb{R}^n$ is chosen in the partitions of index sets $\mathcal{P}$ and $\mathcal{Q}$. Finally, note that whenever $\mathcal{Q}_I^{<}(x^*) = \mathcal{Q}_F^{<}(x^*)$ the results for the infeasible and feasible point partitions coincide.

We now present a result showing that the existence of a saddle point of the Lagrangian function $L(x, \lambda, \mu)$ is a necessary (and, in the case of the feasible partition, sufficient) condition for a point x^* to be Pareto. The observations regarding nonPareto points provide the foundation for the proof of Theorem 3 which follows upon the proof of the main result in [22] and is therefore not repeated here.

THEOREM 3. *If a point* $x^* \in F$ *is a Pareto solution for* (MOP) *then there exist multipliers* $\lambda^* \geqslant 0, \lambda^* \neq 0$ *and* $\mu^* \geqslant 0$ *such that*

$$L(x^*, \lambda^*, \mu) \leqslant L(x^*, \lambda^*, \mu^*) \leqslant L(x, \lambda^*, \mu^*) \tag{13}$$

holds for all $\mu \geqslant 0$ *and for all* $x \in \bar{S}_F^{\leqslant}(x^*)$. *For the feasible point partition the converse holds, too.*

COROLLARY 1. *If* (x^*, λ^*, μ^*) *is a saddle point of* (10) *then* $L(x^*, \lambda^*, \mu^*) = \sum_{i \in \mathcal{Q}^<(x^*)} \lambda_i^* \Phi^i(x^*)$.

Proof. From (13) we have

$$\sup_{\mu \geqslant 0} L(x^*, \lambda^*, \mu) \leqslant L(x^*, \lambda^*, \mu^*) \leqslant \inf_{(x \in \bar{S}_F^{\leqslant}(x^*)} L(x, \lambda^*, \mu^*).$$

We now use (10) to calculate $L(x^*, \lambda^*, 0)$ as $\sum_{(i \in \mathcal{Q}^<(x^*)} \lambda_i^* \Phi^i(x^*)$. Observing that $\mu_j^* \geqslant 0$, for all $j \in \mathcal{P}^<(x^*)$ and $f_j(x^*) \leqslant 0$ for all $j \in \mathcal{P}^<(x^*)$ we conclude

$$\begin{aligned} L(x^*, \lambda^*, 0) = \sum_{i \in \mathcal{Q}^<(x^*)} \lambda_i^* \Phi^i(x^*) &\leqslant \sup_{\mu \geqslant 0} L(x^*, \lambda^*, \mu) \\ &\leqslant L(x^*, \lambda^*, \mu^*) \\ &= \sum_{i \in \mathcal{Q}^<(x^*)} \lambda_i^* \Phi^i(x^*) + \sum_{j \in \mathcal{P}^<(x^*)} \mu_j^* f^j(x^*) \\ &\leqslant \sum_{i \in \mathcal{Q}^<(x^*)} \lambda_i^* \Phi^i(x^*), \end{aligned}$$

which implies the claim. □

For the feasible point partition, we remark that Theorem 3 is of theoretical interest since no constraint qualification, for example such as the Slater condition, is imposed. Nevertheless, Theorem 3 and Corollary 1 seem not to provide useful information if x^* is a Pareto solution. Then $Q_F^{\leq}(x^*)$ is empty and there is no guarantee that $\mathcal{P}_F^{\leq}(x^*)$ is nonempty. However, Theorem 3 becomes of practical value and leads to new results if $\mathcal{Q}^<(x^*)$ is replaced by any nonempty subset $\mathcal{Q}(x^*)$ of $\mathcal{Q}$ so that $\mathcal{Q}^=(x^*)$ in Definition (9) of $\overline{S}^{\leqslant}(x^*)$ is replaced by $Q \backslash Q(x^*)$.

COROLLARY 2. *Let a point* $x^* \in F$ *be a Pareto point for* (MOP) *and assume that* $\mathcal{Q}(x^*) \subset \mathcal{Q}$ *is a nonempty set. Then there exist* $0 \neq \lambda^* \geqslant 0$, $\mu^* \geqslant 0$ *such that the saddle point condition* (13) *and Corollary* 1 *hold for*

$$L(x, \lambda, \mu) = \sum_{i \in \mathcal{Q}(x^*)} \lambda_i \Phi^i(x) + \sum_{j \in \mathcal{P}^<(x^*)} \mu_j f^j(x) \tag{14}$$

for all $\mu \geqslant 0$ and $x \in \tilde{S}^{\leqslant}(x^*)$ *defined as*

$$\tilde{S}^{\leqslant}(x^*) := \{x \in \mathbb{R}^n : \Phi^i(x) \leqslant \Phi^i(x^*) \quad \forall i \in \mathcal{Q} \setminus \mathcal{Q}(x^*)\} \cap \\ \{x \in \mathbb{R}^n : f^j(x) \leqslant 0 \qquad \forall j \in \mathcal{P}^=(x^*)\}. \tag{15}$$

Proof. Follow the proof of necessity in the proof of Theorem 2.4 from [22] and use Corollary 1, replacing $\mathcal{Q}^<(x^*)$ by $\mathcal{Q}(x^*)$. □

COROLLARY 3. *Under the assumptions of Corollary 2 there exists an* $i \in \mathcal{Q}(x^*)$ *such that* $\Phi^i(x^*) \leqslant \Phi^i(x)$ *for all* $x \in \tilde{S}^{\leqslant}(x^*) \cap F$.

Proof. Based on Corollary 2 and using $\mu_j^* f^j(x^*) = 0$ for all $j \in \mathcal{P}^<(x^*)$, as follows from the proof of Corollary 1, the second saddle point inequality (13) for Lagrangian (14) yields

$$\sum_{i \in \mathcal{Q}(x^*)} \lambda_i^* \in \Phi^i(x^*) - \Phi^i(x^*)) - \sum_{j \in \mathcal{P}^<(x^*)} \mu_j^* f^j(x) \leqslant 0 \quad \forall x \in \tilde{S}^{\leqslant}(x^*). \tag{16}$$

Since, by assumption, $x \in \tilde{S}^{\leqslant}(x^*) \cap F$ then $f^j(x) \leqslant 0$, which yields

$$\sum_{j \in \mathcal{P}^<(x^*)} \mu_j^* f^j(x) \leqslant 0$$

in (16). In order that (16) holds it must be that

$$\exists i \in \mathcal{Q}(x^*) : \Phi^i(x^*) \leqslant \Phi^i(x) \quad \text{for all } x \in \tilde{S}^{\leqslant}(x^*) \cap F.$$ □

The results above can have the following interpretation. Given a solution $\hat{x} \in F$, we may not know whether it is a Pareto point. Assuming that it is not, one may want to improve some objective functions that are of special importance without deteriorating the others. In this context, Corollaries 2 and 3 become meaningful. Let $\mathcal{Q}(\hat{x})$ be the set of indices of the objective functions we would like to improve. If the Lagrangian function $L(x, \lambda, \mu)$ does not have a saddle point, i.e. if (13) does not hold for some $\mu \geqslant 0$ and some $x \in \tilde{S}^{\leqslant}(\hat{x})$, then $\hat{x}$ is not Pareto and one can in fact improve the chosen objective functions. On the other hand, if $\hat{x}$ is a Pareto point, then based on Corollary 3, there exists at least one objective function such that its value at $\hat{x}$ is the smallest for all $x \in \tilde{S}^{\leqslant}(\hat{x}) \cap F$, which indicates that at least one of the objective functions chosen cannot be improved and therefore should leave $\mathcal{Q}(\hat{x})$.

Corollary 2 reveals additional information about a Pareto solution examined in the context of a decision situation in which the objective functions of interest are chosen to be in $\mathcal{Q}(x^*)$. As a Pareto solution x^* contributes to the saddle point (x^*, λ^*, μ^*) of function (14), this function measures the cost (to be minimized) of the Pareto point with respect to the criteria of interest and the constraint functions in $\mathcal{P}^{<}(x^*)$ as well as the utility (to be maximized) of that Pareto point with respect to the slack of these constraints. According to Corollary 1, the maximum utility is always constant (equal to $\sum_{i \in \mathcal{Q}(x^*)} \lambda_i^* \Phi^i(x^*)$) and independent of the constraints. Therefore, at a Pareto point the overall slack is always maximized and equal to zero. Viewing the constraints of (MOP) as, for example, resources to be utilized, we may conclude that utility of a Pareto point is independent of the usage of the resources.

THEOREM 4. *Let a point* $x^* \in F$ *be a Pareto point for* (MOP) *and assume that* $\mathcal{Q}(x^*) \subset \mathcal{Q}$ *is a nonempty set. Then* x^* *is a Pareto point of the following multiple objective program*:

$$\begin{aligned} \min \quad & \{\Phi^i(x), i \in \mathcal{Q}(x^*)\} \\ \text{subject to } & f^j(x) \leqslant 0, \quad j \in \mathcal{P}^{<}(x^*) \\ & x \in \tilde{S}^{\leqslant}(x^*) \end{aligned} \qquad ((\text{MOP}(\mathcal{Q}(x^*)))$$

where $\tilde{S}^{\leqslant}(x^*)$ *is defined in* (15).

Proof. Since $x^* \in F$ and Corollary 2 holds, there exist $0 \neq \lambda^* \geqslant 0$, $\mu^* \geqslant 0$ such that the saddle point condition (13) is satisfied for function (14) for all $\mu \geqslant 0$ and $x \in \tilde{S}^{\leqslant}(x^*)$. Using the saddle point optimality conditions of convex single objective programming [2], we get that x^* is an optimal solution of the single objective program

$$\begin{aligned} \min \quad & \sum_{i \in \mathcal{Q}(x^*)} \lambda_i^* \Phi^i(x) \\ \text{subject to } & f^j(x) \leqslant 0, \quad j \in \mathcal{P}^{<}(x^*) \\ & x \in \widetilde{S}^{\leqslant}(x^*). \end{aligned}$$

Applying Geoffrion's result on the weighting scalarization of MOPs [9], we obtain that x^* is a weak Pareto point of the problem (MOP($\mathcal{Q}(x^*)$)). (A point x^* is called a weak Pareto point if there is no other point x such that $\Phi^i(x) < \Phi^i(x^*)$ for all i.)

Suppose now that x^* is a weak Pareto solution but not a Pareto solution for (MOP($\mathcal{Q}(x^*)$)). Then there must exist a point x' feasible for (MOP($\mathcal{Q}(x^*)$)) and some $k \in \mathcal{Q}(x^*)$ such that $\Phi^k(x') < \Phi^k(x^*)$ and also some $i \in \mathcal{Q}(x^*)$ such that $\Phi^i(x') = \Phi^i(x^*)$. Define $\widetilde{\mathcal{Q}}(x^*) := \mathcal{Q}(x^*) \setminus \{i \in \mathcal{Q}(x^*) : \Phi^i(x') = \Phi^i(x^*)\}$. Then, as we proved above, we conclude that x^* is a weak Pareto point for (MOP($\widetilde{\mathcal{Q}}(x^*)$)). Repeating this process, we must finally obtain a set $\widetilde{\mathcal{Q}}(x^*) \neq \emptyset$ and a corresponding point $\tilde{x}'$ such that $\Phi^k(\tilde{x}') < \Phi^k(x^*)$ for all $k \in \widetilde{\mathcal{Q}}(x^*)$. But this contradicts weak Pareto optimality of x^* for (MOP($\widetilde{\mathcal{Q}}(x^*)$)), and therefore x^* must also be a Pareto point of (MOP($\widetilde{\mathcal{Q}}(x^*)$)). $\square$

In the context of Theorem 4 we remark that when all λ_i^* implied by Corollary 2 are positive, x^* is a proper Pareto point of (MOP($\mathcal{Q}(x^*)$)) (in the sense of Geoffrion).

Interestingly, Theorem 4 complements Theorem 1 as they both examine relationships between (MOP) and the reduced MOPs related to the original one. These theorems result in a following corollary:

COROLLARY 4. *Let x^0 be a feasible point of* (MOP) *which is not a Pareto point. If $\bar{x} \in S^{\leqslant}(x^0) \cap F$ is a Pareto point for* (MOP($\mathcal{Q}^{<}(x^0)$)), *then it is also a Pareto point for* (MOP($\mathcal{Q}(x)$)), *where $\mathcal{Q}(\bar{x}) \subset \mathcal{Q}$ is a nonempty set.*

Using (15), (MOP($\mathcal{Q}(x^*)$)) can be written as

$$\begin{aligned} &\min && \{\Phi^i(x), \quad i \in \mathcal{Q}(x^*)\} \\ &\text{subject to} && \Phi^q(x) \leqslant \Phi^q(x^*), \quad q \in \mathcal{Q} \setminus \mathcal{Q}(x^*) \\ &&& x \in F. \end{aligned}$$

Since $\mathcal{Q}(x^*)$ is any nonempty subset of $\mathcal{Q}$, it can be also defined as $\mathcal{Q}(x^*) = \mathcal{Q}^{<}(x^0)$ where $x^0 \in F$ is not a Pareto point for (MOP). Therefore, the problem (MOP($\mathcal{Q}(x^*)$)) then becomes

$$\begin{aligned} &\min && \{\Phi^i(x), i \in \mathcal{Q}^{<}(x^0)\} \\ &\text{subject to} && \Phi^q(x) \leqslant \Phi^q(x^*), \quad q \in \mathcal{Q}^{=}(x^0), \\ &&& x \in F, \end{aligned}$$

which exactly has the structure of the problem (MOP($\mathcal{Q}^{<}(x^0), x^*$)), see Section 2. Clearly, Theorem 2 applies to this special case of the problem (MOP($\mathcal{Q}(x^*)$)) with $\mathcal{Q}(x^*) = Q^{<}(x^0)$ which in this particular case strengthens the result of Theorem 4.

EXAMPLE 1. For the feasible point partition, we illustrate our results with an example extending [12, Example 3.3] by adding another objective function. Consider the following (MOP):

$$\begin{aligned} &\min && \{-x_1 + x_2, x_1^2 + x_2^2, x_2\} \\ &\text{subject to} && f^1(x) = x_1^2 + x_2^2 - 2 \leqslant 0, \\ &&& f^2(x) = -x_1 + x_2^2 - 1 \leqslant 0, \\ &&& f^3(x) = x_1 - x_2 - 1 \leqslant 0, \\ &&& f^4(x) = -x_1 \leqslant 0. \end{aligned} \tag{17}$$

First, choose $x^* = (0,0)$. Then we obtain $\Phi^1(x^*) = \Phi^2(x^*) = \Phi^3(x^*) = 0$. We also get $S_F^{\leqslant}(x^*) = \{x^*\}$ so that $\mathcal{Q}_F^{<}(x^*) = \emptyset$ and therefore x^* is a Pareto point for the original (MOP) (17). Choose now $\mathcal{Q}(x^*) = \{1,2\}$. We then have $\mathcal{P}_F^{<}(x^*) = \{1,2,3\}$ and $\tilde{S}^{\leqslant}(x^*) = \{x \in \mathbb{R}^2 : x_1 \geqslant 0, \ x_2 \leqslant 0\}$. Theorem 4 implies that x^* is a Pareto point for the reduced (MOP) below

$$\begin{array}{ll} \min & \{-x_1+x_2, x_1^2+x_2^2\} \\ \text{subject to} & f^1(x) = x_1^2+x_2^2-2 \leqslant 0, \\ & f^2(x) = -x_1+x_2^2-1 \leqslant 0, \\ & f^3(x) = x_1-x_2-1 \leqslant 0 \\ & x \in \tilde{S}_F^{\leqslant}(x^*), \end{array} \tag{18}$$

whose feasible set can be simplified to $\text{conv}\{(0,0),(1,0),(0,-1)\}$. Checking the saddle point condition of Corollary 2 we have to find $\lambda^* \geqslant 0$, $\lambda^* \neq 0$ and $\mu^* \geqslant 0$ such that

$$\begin{aligned} -2\mu_1-\mu_2-\mu_3 \leqslant & -2\mu_1^*-\mu_2^*-\mu_3^* \\ & \leqslant \lambda_1^*(-x_1+x_2)+\lambda_2^*(x_1^2+x_2^2)+ \\ & \quad +\mu_1^*(x_1^2+x_2^2-2)+\mu_2^*(-x_1+x_2^2-1)+\mu_3^*(x_1-x_2-1) \end{aligned} \tag{19}$$

holds for all $\mu \geqslant 0$ and all $x \in \tilde{S}_F^{\leqslant}(x^*)$. From the first inequality in (19) we get $\mu^* = 0$, as none of the constraints in $\mathcal{P}_F^{<}(x^*)$ is active at x^*. The second inequality is then satisfied choosing, for example, $\lambda_1^* = 0$, $\lambda_2^* = 1$. We remark that it is not possible to find $\lambda_1^* > 0$, $\lambda_2^* > 0$ in (19), and we cannot conclude that x^* is a proper Pareto point for the reduced (MOP). The maximum utility of x^* is quantified as $\sum_{i=1,2} \lambda_i^* \Phi^i(x^*)$.

Consider now the point $x^* = (0.5+0.5\sqrt{3}, -0.5+0.5\sqrt{3})$. Then we obtain $\Phi^1(x^*) = -1$, $\Phi^2(x^*) = 2$, and $\Phi^3(x^*) = -0.5+0.5\sqrt{3}$. We also get $S_F^{\leqslant}(x^*) = \text{conv}\{(0,-1), x^*\}$ so that $\mathcal{Q}_F^{<}(x^*) = \{2,3\}$ and therefore x^* is not a Pareto point of the original (MOP). Choose $\mathcal{Q}(x^*) = \{1\}$. Then $\mathcal{P}_F^{<}(x^*) = \{1,2,4\}$ and $\tilde{S}_F^{\leqslant}(x^*) = \{x \in \mathbb{R}^2 : x_1^2+x_2^2 \leqslant 2, \quad x_2 \leqslant -0.5+0.5\sqrt{3}, x_1-x_2 \leqslant 1\}$. Consider the following reduced problem

$$\begin{array}{ll} \min & \Phi^1(x) = -x_1+x_2 \\ \text{subject to} & f^1(x) = x_1^2+x_2^2-2 \leqslant 0, \\ & f^2(x) = -x_1+x_2^2-1 \leqslant 0, \\ & f^4(x) = -x_1 \leqslant 0, \\ & x \in \tilde{S}_F^{\leqslant}(x^*), \end{array} \tag{20}$$

which is equivalent to

$$\min \quad \{\Phi^1(x) : x \in \text{conv}\{(0,-0.5+0.5\sqrt{3}), x^*, (0,-1)\}\}. \tag{21}$$

Since x^* is not Pareto for the original (MOP) but it is an optimal solution of problem (21), we note that Theorem 4 cannot be strengthened to a necessary and sufficient condition. However, since x^* is not Pareto, we can also apply Theorem 1 and construct the reduced (MOP) below

$$\begin{array}{ll} \min & \{x_1^2+x_2^2, x_2\} \\ \text{subject to} & x \in S_F^{\leqslant}(x^*) = \text{conv}\{(0,-1), x^*\}. \end{array} \tag{22}$$

All Pareto points of (22) are contained in the line segment

$\text{conv}\{(0, -1), (0.5, -0.5)\}$

and according to Theorem 1 they all are Pareto points for the original problem (17).

EXAMPLE 2. We examine the original (MOP) of Example 1 considering the infeasible point partition of index sets. First, for $x^* = (0, 0)$, we observe that the same results can be obtained due to the fact that $S_F^{\leqslant}(x^*) = S_I^{\leqslant}(x^*) = \{x^*\}$.

However, for the point $x^* = (0.5 + 0.5\sqrt{3}, -0.5 + 0.5\sqrt{3})$ these two sets differ. We have

$$S_F^{\leqslant}(x^*) = \{x \in \mathbb{R}^2 : -x_1 + x_2 \leqslant -1, x_1^2 + x_2^2 \leqslant 2, x_2 - 0.5 + 0.5\sqrt{3}\}.$$

The set $S_F^{\leqslant}(x^*)$ in Example 1 is only part of the boundary of $S_I^{\leqslant}(x^*)$. We also have $\mathcal{Q}_I^{<}(x^*) = \mathcal{Q}$, and $\mathcal{P}_I^{<}(x^*) = \{1, 2, 4\}$. Therefore from Theorem 1 we obtain that all Pareto points of the multiple objective program

$$\begin{array}{ll} \min & \{-x_1 + x_2, x_1^2 + x_2^2, x_2\} \\ \text{subject to} & x \in S_I^{\leqslant}(x^*) \end{array} \tag{23}$$

which are feasible for (MOP) are also Pareto points of (MOP).

4. Nonconvex Problems

In this section we drop the convexity assumptions of the previous section, consider the general (nonconvex) (MOP) and discuss again the relationships between Pareto points and saddle points using a particular augmented Lagrangian. Before we state main results, we first make certain assumptions about single objective programs related to MOPs of interest and examine the conditions under which those assumptions hold. As the linear Lagrangian (10) cannot be used for nonconvex problems, we associate with (CC(MOP, x^*)) the augmented Lagrangian (suggested in [14] and investigated in [4] and [13]) defined for $x \in F$, $y \in \mathbb{R}^{\mathcal{Q}}$ and $r > 0$:

$$\begin{aligned} L_a(x, y, r) = \sum_{q \in \mathcal{Q}} \Phi^q(x) + \sum_{q \in \mathcal{Q}} \Big[y_q \max\Big\{\Phi^q(x) - \Phi^q(x^*), \frac{-y_q}{2r}\Big\} \\ + r\Big(\max\Big\{\Phi^q(x) - \Phi^q(x^*), \frac{-y_q}{2r}\Big\}\Big)^2\Big], \end{aligned} \tag{24}$$

where $x^* \in F$.

Following Rockafellar, see [13], we make the following assumptions about (CC(MOP,x^*)).

ASSUMPTION 1. Let (CC(MOP, x^*)) satisfy the quadratic growth condition (QGC), i.e., there exists an $r \geqslant 0$ such that $L_a(x, 0, r)$ is bounded below as a function of x, $x \in F$.

The QGC is certainly satisfied if, for some $r \geqslant 0$, all the functions Φ^q are bounded below on F and thus in particular, if F is compact and all the functions are lower semicontinuous on F. Such an assumption is in order, since it guarantees the existence of Pareto solutions, see e.g., [16]. For more details about the quadratic growth condition we refer to Rockafellar, [13].

ASSUMPTION 2. Let (CC(MOP,x^*)) be (lower) stable of degree 2 (SoD2), i.e. there exist

1. an open neighbourhood N of the origin in $\mathbb{R}^Q$, and
2. a function $\pi : N \to \mathbb{R}^1$ of class C^2 (i.e. twice continuously differentiable) such that

$$p(u) \geqslant \pi(u) \quad \text{for all } u \in N$$

and

$$p(0) = \pi(0),$$

where $p(u) : \mathbb{R}^Q \to \mathbb{R}^1$ is the perturbation function associated with (CC(MOP, x^*)) and defined as

$$p(u) = \min_{x \in F} \left\{ \sum_{q \in Q} \Phi^q(x) : \Phi^q(x) - \Phi^q(x^*) \leqslant u_q; q \in Q, u \in \mathbb{R}^Q \right\}.$$

We emphasize that Assumption 1 is rather technical and not constraining while Assumption 2 is stronger and related to the curvature of the original objective functions, which should allow that the perturbation function $p(u)$ of (CC(MOP,x^*)) be supported by a C^2 function π. The following lemmas examine when Assumption 2 holds.

LEMMA 1. *Let $x^* \in F$. If the single objective problems $(P_q), q \in Q$*

$$\begin{array}{ll} \min & \Phi^q(x) \\ \text{subject to} & \Phi^i(x) \leqslant \Phi^i(x^*), \quad i = 1, \ldots, Q \\ x \in F & \end{array} \qquad (P_q)$$

are SoD2 then there exist a neighbourhood N of the origin and a function π of class C^2 such that the perturbation function $p(u)$ of (CC(MOP,x^*)) *is bounded below by π.*

Proof. Let $N_1, \ldots, N_q \subset \mathbb{R}^Q$ and $\pi_1, \ldots, \pi_Q$ be the neighborhoods and functions that exist according to the SoD2 condition for problems $(P_q), q = 1, \ldots, Q$. Define the open neighborhood N of the origin in $\mathbb{R}^Q$,

$$N := \cap_{q=1}^{Q} N_q$$

and the function $\pi : N \to \mathbb{R}^1$ of class C^2 as

$$\pi(u) := \sum_{q \in Q} \pi_q(u).$$

Then for each $u \in N$ we have

$$\begin{aligned} p(u) &= \min_{x\in F}\left\{\sum_{q\in Q}\Phi^q(x):\Phi^i(x)\leqslant\Phi^i(x^*)+u_i;\ i=1,\ldots,Q\right\} \\ &\geqslant \sum_{q\in Q}(\min_{x\in F}\{\Phi^q(x):\Phi^i(x)\leqslant\Phi^i(x^*)+u_i;\ i=1,\ldots,Q\}) \qquad (25) \\ &= \sum_{q\in Q}p_q(u)\geqslant\sum_{q\in Q}\pi_q(u)=\pi(u). \end{aligned}$$

The first equality in (25) is the definition of $p(u)$, the first inequality is obvious, the second equality is the definition of $p_q(u)$, the second inequality is the SoD2 assumption for (P_q), and the final equality is the definition of π.

□

Due to Lemma 1, a sufficient condition for (CC(MOP,x^*)) to be SoD2 is that the single objective problems (P_q) are SoD2 and additionally satisfy

$$p(0)=\sum_{q\in Q}\pi_q(0)=\sum_{q\in Q}p_q(0), \qquad (26)$$

where p, p_q and π_q are defined as in the lemma. More important for further analysis is the following lemma which reveals that this condition is satisfied if x^* in Lemma 1 is a Pareto point of (MOP).

LEMMA 2. *If, under the assumptions of Lemma* 1, x^* *is a Pareto point of* (MOP), *then* (CC(MOP, x^*)) *is* SoD2.

Proof. From Lemma 1 and its proof we know that the perturbation function $p(u)$ of (CC(MOP, x^*)) is bounded below by the C^2 function $\pi(u)=\sum_{q\in Q}\pi_q(u)$. Now consider the constraints $\Phi^i(x)\leqslant\Phi^i(x^*)$, $i=1,\ldots,Q$, of a single objective problem (P_q). If $x\in F$ is feasible for (P_q) then $x\in\cap_{i=1}^{Q}L^i_{\leqslant}(x^*)$. Since x^* is a Pareto point of (MOP), we can apply (2) to get that

$$x\in\bigcap_{i=1}^{Q}L^i_{=}(x^*).$$

Therefore $\Phi^i(x)=\Phi^i(x^*)$ for all $i=1,\ldots,Q$ and all $x\in F$ feasible for (P_q). Consequently, $\sum_{q\in Q}\Phi^q(x)$ is constant over the (identical) feasible set of each (P_q), which is also the feasible set of (CC(MOP,x^*)). Thus (26) follows and the lemma is proved. □

We now relate a Pareto point of (MOP) to a saddle point of the augmented Lagrangian.

THEOREM 5. *Let* (CC(MOP,x^*)) *satisfy* QGC *and be* SoD2. *Then* $x^*\in F$ *is a Pareto point for* (MOP) *if and only if there exist* (y^*,r^*), $r^*>0$ *such that*

$$L_a(x^*, y, r) \leqslant L_a(x^*, y^*, r^*) \leqslant L_a(x, y^*, r^*) \tag{27}$$

holds for all $x \in F$ *and* $y \in \mathbb{R}^Q$, $r > 0$.

Proof. According to the Charnes and Cooper result, $x^* \in F$ is a Pareto solution of (MOP) if and only if it is an optimal solution of (CC(MOP,x^*)). Applying to this problem the results by Rockafellar [13, Corollary 5.2], we obtain the desired result. □

Using [1, Theorem 2.2], we can similarly show that if x^* is an isolated local solution of (CC(MOP,x^*)) satisfying the standard second order sufficiency conditions for optimality with strict complementarity, then there exists a neighborhood N of x^* such that

$$\min_{x \in N} L_a(x, y^*, r^*) = L_a(x^*, y^*, r^*) = \max_{y \in \mathbb{R}^Q} L_a(x^*, y, r^*),$$

where y^* is the Lagrange multiplier vector and r^* is sufficiently large.

In the remaining part of this section we turn our attention to the feasible point partition and reformulate (CC(MOP($\mathcal{Q}_F^<(x^*)), \hat{x}$)) in order to proceed with other results.

Because $\hat{x} \in S_F^{\leqslant}(x^*)$ we have

$$\Phi^q(\hat{x}) \leqslant \Phi^q(x^*) \quad \forall q \in \mathcal{Q}. \tag{28}$$

From the definition of (CC(MOP($\mathcal{Q}_F^<(x^*)), \hat{x}$)) we get

$$\left.\begin{array}{l} \Phi^q(x) - \Phi^q(\hat{x}) \leqslant 0 \\ \Phi^q(x) - \Phi^q(x^*) \leqslant 0 \end{array}\right\} \quad \forall q \in \mathcal{Q}_F^<(x^*) \tag{29}$$

and

$$\Phi^q(x) - \Phi^q(x^*) \leqslant 0 \qquad \forall q \in \mathcal{Q}_F^=(x^*). \tag{30}$$

With (28), inequalities (29) and (30) can be written as

$$\Phi^q(x) - \Phi^q(\hat{x}) \leqslant 0 \quad \forall q \in \mathcal{Q}_F^<(x^*)$$
$$\Phi^q(x) - \Phi^q(x^*) \leqslant 0 \quad \forall q \in \mathcal{Q}_F^=(x^*).$$

Now (CC(MOP($\mathcal{Q}_F^<(x^*)), \hat{x}$)) becomes

$$\begin{array}{ll} \min & \displaystyle\sum_{q \in \mathcal{Q}_F^<(x^*)} \Phi^q(x) \\ \text{subject to} & \Phi^q(x) - \Phi^q(\hat{x}) \leqslant 0 \quad \forall q \in \mathcal{Q}_F^<(x^*), \\ & \Phi^q(x) - \Phi^q(x^*) \leqslant 0 \quad \forall q \in \mathcal{Q}_F^=(x^*), \\ & x \in F. \end{array} \tag{31}$$

Let $p^s(u)$ be the perturbation function associated with problem (31):

$$p^s(u) = \min_{x \in F} \left\{ \sum_{q \in \mathcal{Q}_F^<(x^*)} \Phi^q(x) : \right.$$
$$\left. \begin{bmatrix} \Phi^q(x) - \Phi^q(\hat{x}) & \forall q \in \mathcal{Q}_F^<(x^*) \\ \Phi^q(x) - \Phi^q(x^*) & \forall q \in \mathcal{Q}_F^=(x^*) \end{bmatrix} \leqslant u, u \in \mathbb{R}^{\mathcal{Q}} \right\}. \tag{32}$$

We show that problem (31) is SoD2 if the same is true for the original problem (CC(MOP,$\hat{x}$)).

LEMMA 3. *Consider $x^* \in F$ that is not a Pareto point for* (MOP). *Then for all $\hat{x} \in S_F^{\leqslant}(x^*)$ such that* (CC(MOP,$\hat{x}$)) *is* SoD2, (CC(MOP($\mathcal{Q}_F^<(x^*)$), $\hat{x}$)) *is also* SoD2.

Proof. Let $p(u)$ and $p^s(u)$ be the perturbation functions associated with scalarizations (CC(MOP,$\hat{x}$)) and (CC(MOP($\mathcal{Q}_F^<(x^*)$), $\hat{x}$)), respectively. We define an appropriate function $\pi^s(u)$ of class C^2 and first show that $p^s(u) \geqslant \pi^s(u)$.

In the second part we show that also $p^s(0) = \pi^s(0)$.

Let

$$\pi^s(u) := \pi(u) - \sum_{q \in \mathcal{Q}_F^=(x^*)} (\Phi^q(\hat{x}) + u_q) \tag{33}$$

where $\pi(u)$ is defined as in Assumption 2 for (CC(MOP,$\hat{x}$)) and

$$p(u) \geqslant \pi(u) \quad \forall u \in N. \tag{34}$$

From (33) and (34) we get

$$p(u) - \sum_{q \in \mathcal{Q}_F^=(x^*)} (\Phi^q(\hat{x}) + u_q) \geqslant \pi^s(u). \tag{35}$$

Now we shall show that

$$p(u) - \sum_{q \in \mathcal{Q}_F^=(x^*)} (\Phi^q(\hat{x}) + u_q) \leqslant p^s(u). \tag{36}$$

By definition,

$$p(u) = \min_{x \in F} \left\{ \sum_{q \in \mathcal{Q}} \Phi^q(x) : \Phi^q(x) - \Phi^q(\hat{x}) \leqslant u_q \quad \forall q \in \mathcal{Q} \right\}$$
$$= \min_{x \in F} \left\{ \sum_{q \in \mathcal{Q}_F^<(x^*)} \Phi^q(x) + \sum_{q \in \mathcal{Q}_F^=(x^*)} \Phi^q(x) : \begin{bmatrix} \Phi^q(x) - \Phi^q(\hat{x}) & \forall q \in \mathcal{Q}_F^<(x^*) \\ \Phi^q(x) - \Phi^q(\hat{x}) & \forall q \in \mathcal{Q}_F^=(x^*) \end{bmatrix} \leqslant u \right\},$$

since $\mathcal{Q} = \mathcal{Q}_F^<(x^*) \cup \mathcal{Q}_F^=(x^*)$ (see (3) and (4)).

Observe that

$$\Phi^q(\hat{x}) = \Phi^q(x^*) \quad \text{for all } q \in \mathcal{Q}_F^=(x^*).$$

Therefore

$$
\begin{aligned}
p(u) &= \min_{x\in F}\left\{\sum_{q\in\mathcal{Q}_F^<(x^*)}\Phi^q(x)+\sum_{q\in\mathcal{Q}_F^=(x^*)}\Phi^q(x):\begin{bmatrix}\Phi^q(x)-\Phi^q(\hat{x}) & \forall q\in\mathcal{Q}_F^<(x^*)\\ \Phi^q(x)-\Phi^q(x^*) & \forall q\in\mathcal{Q}_F^=(x^*)\end{bmatrix}\leqslant u\right\}\\
&\leqslant \min_{x\in F}\left\{\sum_{q\in\mathcal{Q}_F^<(x^*)}\Phi^q(x)+\sum_{q\in\mathcal{Q}_F^=(x^*)}(\Phi^q(\hat{x})+u_q):\begin{bmatrix}\Phi^q(x)-\Phi^q(\hat{x}) & \forall q\in\mathcal{Q}_F^<(x^*)\\ \Phi^q(x)-\Phi^q(x^*) & \forall q\in\mathcal{Q}_F^=(x^*)\end{bmatrix}\leqslant u\right\}\\
&= \min_{x\in F}\left\{\sum_{q\in\mathcal{Q}_F^<(x^*)}\Phi^q(x):\begin{bmatrix}\Phi^q(x)-\Phi^q(\hat{x}) & \forall q\in\mathcal{Q}_F^<(x^*)\\ \Phi^q(x)-\Phi^q(x^*) & \forall q\in\mathcal{Q}_F^=(x^*)\end{bmatrix}\leqslant u\right\}\\
&\quad+\sum_{q\in\mathcal{Q}_F^=(x^*)}(\Phi^q(\hat{x})+u_q)\\
&= p^s(u)+\sum_{q\in\mathcal{Q}_F^=(x^*)}(\Phi^q(\hat{x})+u_q), \qquad (37)
\end{aligned}
$$

where $p^s(u)$ is the perturbation function defined in (32). The inequality above results from the condition $\Phi^q(x)-\Phi^q(x^*)\leqslant u_q\ \forall q\in\mathcal{Q}_F^=(x^*)$, after which the expression under the second sum becomes constant and the minimization is carried out with respect to the first sum. Clearly, (37) proves (36).

Now we shall show that $p^s(0)=\pi^s(0)$.

$$
\begin{aligned}
p(0) &= \min_{x\in F}\left\{\sum_{q\in\mathcal{Q}_F^<(x^*)}\Phi^q(x)+\sum_{q\in\mathcal{Q}_F^=(x^*)}\Phi^q(x):\begin{bmatrix}\Phi^q(x)-\Phi^q(\hat{x}) & \forall q\in\mathcal{Q}_F^<(x^*)\\ \Phi^q(x)-\Phi^q(\hat{x}) & \forall q\in\mathcal{Q}_F^=(x^*)\end{bmatrix}\leqslant 0\right\}\\
&= \min_{x\in S_F^{\leqslant}(x^*)}\left\{\sum_{q\in\mathcal{Q}_F^<(x^*)}\Phi^q(x):\Phi^q(x)-\Phi^q(\hat{x})\leqslant 0\quad\forall q\in\mathcal{Q}_F^<(x^*)\right\}\\
&\quad+\sum_{q\in\mathcal{Q}_F^=(x^*)}\Phi^q(\hat{x})\\
&= \min_{x\in F}\left\{\sum_{q\in\mathcal{Q}_F^<(x^*)}\Phi^q(x):\begin{bmatrix}\Phi^q(x)-\Phi^q(\hat{x}) & \forall q\in\mathcal{Q}_F^<(x^*)\\ \Phi^q(x)-\Phi^q(x^*) & \forall q\in\mathcal{Q}_F^=(x^*)\end{bmatrix}\leqslant 0\right\}\\
&\quad+\sum_{q\in\mathcal{Q}_F^=(x^*)}\Phi^q(\hat{x})\\
&= p^s(0)+\sum_{q\in\mathcal{Q}_F^=(x^*)}\Phi^q(\hat{x}).
\end{aligned}
$$

The second equality follows from the definition of $S_F^{\leqslant}(x^*)$ (see (1)) and $\Phi^q(x)=\Phi^q(x^*)$ for all $x\in S_F^{\leqslant}(x^*)$ and all $q\in\mathcal{Q}_F^=(x^*)$. Now we apply the definition of $S_F^{\leqslant}(x^*)$ again and get the third equality.

Since $p(0)=\pi(0)=\pi^s(0)+\sum_{q\in\mathcal{Q}_F^=(x^*)}\Phi^q(\hat{x})$ we obtain the desired result. □

We are now in the position to present another result on the existence of Pareto-related saddle points. We emphasize that the arguments of Lemma

1 and Lemma 3 show that no stronger assumptions are required for (MOP) than those introduced by Rockafellar [13] for single objective programs.

THEOREM 6. *Let* $x^* \in F$ *be a point which is not a Pareto solution for* (MOP). *Let* $\hat{x} \in S_F^{\leqslant}(x^*)$ *and let* (CC(MOP,$\hat{x}$)) *satisfy the* QGC *and be* SoD2. *If* $(\hat{x}, \hat{y}, \hat{r})$ *is a saddle point of the augmented Lagrangian function associated with scalarization* $(\mathrm{CC}(\mathrm{MOP}(\mathcal{Q}_F^{<}(x^*)), \hat{x}))$, *then there exist* $(y^*, r^*), r^* > 0$, *such that* $(\hat{x}, y^*, r^*)$ *is a saddle point of the augmented Lagrangian function associated with* (CC(MOP,$\hat{x}$)), *and* $\hat{x}$ *is Pareto for* (MOP).

Proof. Since problem (CC(MOP,$\hat{x}$)) satisfies the quadratic growth condition so does $(\mathrm{CC}(\mathrm{MOP}(\mathcal{Q}_F^{<}(x^*), \hat{x})))$. Furthermore, from Lemma 3, we have that problem $(\mathrm{CC}(\mathrm{MOP}(\mathcal{Q}_F^{<}(x^*), \hat{x})))$ is SoD2. From Theorem 5, $\hat{x}$ is a Pareto point for $(\mathrm{MOP}(\mathcal{Q}_F^{\leqslant}(x^*))$, and from Theorem 1, $\hat{x}$ is also Pareto for (MOP). Therefore, due to Theorem 5 again, the result follows. □

EXAMPLE 3. We illustrate Theorem 6 with an example. Consider the following nonconvex (MOP):

$$
\begin{array}{ll}
\min & \{\Phi^1, \Phi^2, \Phi^3\} \\
\text{subject to} & f^1(x) = -(x_1 - 1)^2 + x_2 - 2 \leqslant 0 \\
& f^2(x) = x_1 - x_2^2 - 1 \leqslant 0 \\
& f^3(x) = x_1 \leqslant 2 \\
& f^4(x) = -x_1 \leqslant 0 \\
& f^5(x) = x_2 \leqslant 3 \\
& f^6(x) = -x_2 \leqslant 0
\end{array}
\tag{38}
$$

where the objective functions Φ^1,Φ^2,Φ^3 are defined as follows:

$$
\Phi^1(x) = \begin{cases} -x_1 & \text{if } x_1 \leqslant 1.5 \\ -1.5 & \text{otherwise,} \end{cases}
$$

$$
\Phi^2(x) = \begin{cases} (x_1 - 1)^2 - x_2 + 2 & \text{if } 4(x_1 - 1.5)^2 + 4(x_2 - 2.55)^2 > 1, \\ 0 & \text{otherwise,} \end{cases}
$$

$$
\Phi^3(x) = \begin{cases} -(x_1 - 1)^2 - (x_2 - 5)^2 + 100 & \text{if } 4(x_1 - 1.5)^2 + 4(x_2 - 2.55)^2 > 1 \\ 0 & \text{otherwise.} \end{cases}
$$

Choose $x^* = (0, 3)$. Then we evaluate $[\Phi^1(x^*), \Phi^2(x^*), \Phi^3(x^*)] = [0, 0, 95]$. We have

$$
\begin{aligned}
S_F^{\leqslant}(x^*) = &\{x \in \mathbb{R}^2 : 0 \leqslant x_1 \leqslant 2,\ (x_1 - 1)^2 - x_2 + 2 = 0\} \cup \\
&\{x \in \mathbb{R}^2 : 4(x_1 - 1.5)^2 + 4(x_2 - 2.25)^2 \leqslant 1, -(x_1 - 1)2 + x_2 - 2 \leqslant 0\}.
\end{aligned}
$$

We also obtain $\mathcal{Q}_F^{=}(x^*) = \{2\}$ and $\mathcal{Q}_F^{<}(x^*) = \{1, 3\}$ and the smaller problem (MOP($\mathcal{Q}_F^{<}(x^*)$)) below:

$$\begin{array}{ll} \min & \{\Phi^1, \Phi^3\} \\ \text{subject to} & x \in S_F^{\leqslant}(x^*) \end{array} . \tag{39}$$

Choosing $\hat{x} - (1.5, 2.25)$ we can construct (CC(MOP($\mathcal{Q}_{\hat{F}}^{<}(x^*), \hat{x}$))) and verify the theorem. Observe that all feasible points of this single objective problem have the same objective value.

For another example of the application of the augmented Lagrangian to nonconvex MOPs the reader is referred to [23].

5. Conclusion

In this paper, Pareto solutions of MOPs are related to saddle points of some Lagrangian-type scalarizing functions. The common foundation for all results is determined by Charnes and Cooper's scalarization of MOPs as well as by the partitions of the set of objective functions and the set of constraints used in this paper. Saddle point characterizations of Pareto points for convex and nonconvex MOPs are derived. For convex MOPs, an interpretation of the saddle point condition in decision making is included. For nonconvex MOPs, it is shown that finding Pareto solutions is equivalent to finding saddle points of the augmented Lagrangian function L_a, which, to the authors' knowledge, had not been known before.

References

1. Arrow, K.J., Gould, P.J. and Howe, S.M. (1973), A general saddle point result for constrained optimization. *Mathematical Programming* 5, 225–234.
2. Bazaraa, M.S., Sherali, H.D. and Shetty, C.M. (1993), *Nonlinear Programming: Theory and Algorithms*. John Wiley, New York.
3. Breckner, W.W., Sekatzek, M. and Tammer, C. (2001), Approximate saddle point assertions for a general class of approximation problems. In: Lassonde, M. (ed.), *Approximation, Optimization and Mathematical Economics*, pp. 71–80. Physica, Heidelberg.
4. Buys, J.D. *Dual Algorithms for Constrained Optimization*. Ph.D. thesis, University of Leiden.
5. Charnes, A. and Cooper, W. (1961), *Management Models and Industrial Applications of Linear Programming*. John Wiley and Sons, New York.
6. Dutta J. and Vetrivel, V. (2001), On approximate minima in vector optimization. *Numerical Functional Analysis and Optimization* 22(7–8), 845–859.
7. Ehrgott, M. (1997), *Multiple Criteria Optimization – Classification and Methodology*. Shaker Verlag, Aachen. Ph.D. Dissertation, University of Kaiserslautern.
8. Ehrgott, M., Hamacher, H.W., Klamroth, K., Nickel, S., Schöbel, A. and Wiecek M.M. (1997), A note on the equivalence of balance points and Pareto solutions in multiple-objective programming. *Journal of Optimization Theory and Applications* 92(1), 209–212.

9. Geoffrion, A.M. (1968), Proper efficiency and the theory of vector maximization. *Journal of Mathematical Analysis and Applications*, 22, 618–630.
10. Huang, X.X. and Yang, X.Q. (2001), Duality and exact penalization for vector optimization via augmented lagrangian. *Journal of Optimization Theory and Applications* 111(3), 615–640.
11. Iacob, P. (1986), Saddle point duality theorems for Pareto optimization. *L'Analyse Numerique et la Theorie de l' Approximation* 15(1), 37–40.
12. Nerali, L. and Zlobec, S. (1996), LFS functions in multi-objective programming. *Applications of Mathematics* 41(4), 347–366.
13. Rockafellar, R.T. (1974), Augmented Lagrange multiplier functions and duality in nonconvex programming. *SIAM Journal on Control* 12(2), 268–285.
14. Rockafellar, T.J. (1971), New applications of duality in convex programming. In *Proceedings of the 4th Conference on Probability*, Brasov, Romania.
15. Rong, W.D. and Wu, Y.N. (2000), ϵ-weak minimal solutions of vector optimization problems with set-valued maps. *Journal of Optimization Theory and Applications* 106(3), 569–579.
16. Sawaragi, Y., Nakayama, H., and Tanino, T. (1985), *Theory of Multiobjective Optimization*. Academic Press, Orlando.
17. Singh, C., Bhatia, D. and Rueda, N. (1996), Duality in nonlinear multiobjective programming using augmented Lagrangian functions. *Journal of Optimization Theory and Applications* 88(3), 659–670.
18. TenHuisen, M.L. and Wiecek, M.M. (1994), Vector optimization and generalized Lagrangian duality. *Annals of Operations Research* 51, 15–32.
19. TenHuisen, M.L. and Wiecek, M.M. (1997), Efficiency and solution approaches to bicriteria nonconvex programs. *Journal of Global Optimization* 11(3), 225–251.
20. TenHuisen M.L. and Wiecek, M.M. (1998), An augmented Lagrangian scalarization for multiple objective programming. In: Caballero, R., Ruiz, F. and Steuer, R.E. (ed.), *Advances in Multiple Objective and Goal Programming*: *Proceedings of the MOPGP'96 Conference*, *Malaga*, *Spain*, volume 455 of *Lecture Notes in Economics and Mathematical Systems*, pp. 151–159. Springer Verlag, Berlin.
21. Valyi, I. (1987), Approximate saddle-point theorems in vector optimization. *Journal of Optimization Theory and Applications* 55(3), 435–448.
22. van Rooyen, M., Zhou, X. and Zlobec, S. (1994), A saddle-point characterization of Pareto optima. *Mathematical Programming* 67(1), 77–88.
23. Wiecek, M.M., Chen W. and Zhang J. (2001), Piecewise quadratic approximation of the non-dominated set for bi-criteria programs. *Journal of Multi-Criteria Decision Analysis* 10, 35–47.
24. Zlobec, S. (1984), Two characterizations of Pareto minima in convex multicriteria optimization. *Aplikace Matematiky* 29(5), 342–349.